REMARKS/ARGUMENTS

This Amendment includes a request that this application be furthered considered as an RCE application.

This Amendment includes a Change of Address for the Attorney of Record.

Status of the Application:

Status of the Claims: Claims 1 to 21 have been presented for examination. Claims 1 to 21, all the pending claims, are rejected.

Claims 1 to 19 and 21 are rejected under 35 USC 103(a) as being unpatentable over the combined teachings of Robb et al., Mick et al., Kenet et al. and Gerstenberger.

Claims 1 to 21 are objected to because of noted informalities.

Claims 1 to 21 are rejected under 35 USC 112, second paragraph, because of noted indefiniteness.

Claims 1, 6, 7 and 15 are rejected under 35 USC 103(a) as being unpatentable over Robb et al. in view of Mick et al.

Claims 2 and 3 are rejected under 35 USC 103(a) as being unpatentable over Robb et al. and Mick et al. and further in view of Gerstenberger.

Claim 4 is rejected under 35 USC 103(a) as being unpatentable over Robb et al. and Mick et al. and further in view of Russ.

Claim 5 is rejected under 35 USC 103(a) as being unpatentable over Robb et al. and Mick et al. and further in view of Pratt.

Claims 8 to 14 are rejected under 35 USC 103(a) as being unpatentable over Robb et al. and Mick et al and further in view of Kenet et al.

Claims 16 to 19 and 21 are rejected under 35 USC 103(a) as being unpatentable over Robb et al.

Claims 19 and 20 are rejected under 35 USC 103(a) as being unpatentable over Robb et al. and further in view of Nields.

Status of the Declaration: The Declaration filed October 26, 1999 is acceptable.

Status of the Description: The objection to the description has been withdrawn.

Status of the Drawings: The drawings are objected to by the Draftsperson. As noted by the Applicant in the Amendment of October 26, 2001, new drawings will be submitted upon allowance of any claim or an indication of allowable subject matter. The Examiner has not required satisfying the drawing correction.

Status of the priority claim: The Examiner has acknowledged receipt of a certified copy of the application under 35 USC 119(a)-(d) and 365 and 371.

Status of the Information Disclosure Statement: The Examiner has acknowledged receipt of the Information Disclosure Statement filed by the applicant; however, the Examiner notes that: (1) a concise explanation of a cited non-English document (FR 91 15308) has not been submitted and (2) a copy of the non-patent literature (Serra et al.) has not been submitted.

A concise explanation of FR 91 15308 will be provided in due course. Enclosed herewith isa copy of Serra et al. The applicant notes that when the Information Disclosure Statement was filed on October 26, 2001, there was then no requirement for a concise explanation of a cited non-English document.

Extension: Enclosed herewith is a Petition to Extend the term for response by month to December 3, 2004.

Objection to claims 1 to 21 because of noted informalities

The noted objections have been corrected by deleting the step identifiers and the corresponding periods. Claim 7, line 7, have been amended as proposed.

Rejection of claims 1 to 21 under 35 USC 112, second paragraph, because of noted indefiniteness.

The rejection of claims 1 and 7, "step c" is traversed. The language recited in claims 1 and 7, "step c" is correct and definite as recited and is in accordance with the Detailed Description as filed.

The rejection of claims 1 and 7, "step d" is traversed. The claim feature recites a matching of a "generated target window ... <u>around</u> the selected target pixels." (emphasis added). The claim feature is correct and definite and is in accordance with the Detailed Description as filed.

Claims 1, 7 and 16 have been amended to provide antecedent basis for the noted indefiniteness.

Rejection of claims 1 to 21 under 35 USC 103(a) as being unpatentable over the combined or individual teachings of Robb et al., Mick et al., Kenet et al., Russ, Pratt and Gerstenberger.

The rejection of claims 1 to 19 and 21 under 35 USC 103(a) as being unpatentable over the combined or individual teachings of Robb et al., Mick et al., Kenet et al, Russ, Pratt and Gerstenberger is traversed for the same reasons as presented in the Amendment filed October 26, 2001.

The Court of Appeals for the Federal Circuit has repeatedly held that a rejection under 35 USC 103 must satisfy the criteria of *Graham vs. Deere*. The rejection has failed to satisfy these criteria, particularly of how the features and functions recited in the claims can be found in the cited reference and how any differences between the features and functions are obvious to one skilled in the art. It is not sufficient to merely conclude that the one skilled in the art *could* modify or adapt the cited prior art to therefore read on the features and functions recited in the claims. The Examiner must establish that one skilled in the art *would* modify or adapt the cited prior art and demonstrate the motivation or nexus of the prior art to the features and functions recited in the claims. The Examiner has alleged that the claimed invention is obvious based on a combination of prior art references. It is improper hindsight merely to "pick and choose" features from the prior art references to "arrive" at the invention as claimed. There must be a non-speculative and objective suggestion or motivation in one or more of the cited prior art to combine the features of the prior art in the manner recited in the invention as claimed. A prima facie case of unpatentability is not established by speculating that any claim feature is found in the cited prior art simply because the word "comprising" is recited in the claim. Without this suggestion or motivation or non-speculative indication in the cited prior art the invention as claimed is unobvious and patentable in accordance with 35 USC 103.

Claim 1 has been amended to recite with greater particularity and definiteness the correlation feature that together with other features recited in claim 1, i.e., the subject matter as a whole, is not disclosed in Robb et al. or Mick et al. or Gerstenberger or Kenet et al. or Pratt or Russ. It should be noted that the cited US patent to Gerstenberger corresponds to WO92/06444 cited in the Information Disclosure Statement. in turn, WO92/06444 was cited in the International Search Report as Category A, i.e., only of "technological interest".

Claims 2 to 6 are each dependent on claim 1 and are patentable over the cited references for the same reasons as claim 1.

Rejection of claims 19 and 20 under 35 USC 103(a) as being unpatentable over Robb et al. and further in view of Nields.

The rejection of claims 19 or 20 (or any claims) to include Nields is traversed. Nields is not qualified prior art as noted by the Examiner in paragraph 4 of the Detailed Action of the Office communication of 06/03/2004. Any rejection of any claim based on Nields is moot and should be withdrawn.

In view of the amendments to the claims, in combination with the discussions earlier regarding the rejections under 35 U.S.C. §103(a), Applicant respectfully requests reconsideration and withdrawal of all rejections under 35 U.S.C. §103(a) and 35 U.S.C. §112, second paragraph, which Applicant considers to be traversed. The Applicant respectfully requests that the Examiner reconsider and withdraw all of these rejections and respectfully requests a timely Notice of Allowance with respect to claims 1 to 21.

The Commissioner is hereby authorized to charge any additional fees that may be required for this amendment, or credit any overpayment, to Deposit Account No. 09-0470.

In the event that an extension of time is required, or may be required in addition to that requested in a petition for extension of time, the Commissioner is requested to grant a petition for that extension of time that is required to make this response timely and is hereby authorized to charge any fee for such an extension of time or credit any overpayment for an extension of time to the above identified Deposit Account.

Respectfully submitted,

MULLER ET AL.

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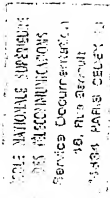# Image Analysis and Mathematical Morphology

Volume 2: Theoretical Advances

Edited by

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1

Mathematical Morphology for Complete Lattices

J. SERRA

1.1 BASIC PROPERTIES OF LATTICES

Let us look briefly at the basic properties and definitions of lattices. The reader will find a more complete study in Birkhoff (1983) and Dubreil and M.L. Dubreil-Jacotin (1964). A sup (resp. inf) semilattice is an ordered set $\mathscr{P}$ in which any two elements X_1 and X_2 have a smallest majorant (resp. greatest minorant) $X_1 \vee X_2$ (resp. $X_1 \wedge X_2$) called the upper bound or sup (resp. lower bound or inf). The terms sup and inf are abbreviations for the Latin words supremum and infimum. The semilattice is complete if each family of elements $X_i \in \mathscr{P}$, finite or not, has an upper bound (resp. lower bound), thus implying the existence of a greatest element (resp. least element) called the universal element U (resp. null element $\emptyset$).

A set that is both a sup and an inf semilattice is called a lattice. If the semi-lattices are complete then so is the lattice. Any complete semilattice with universal and null elements is a complete lattice. In a lattice any logical consequence of a choice of ordering remains true when we commute the symbols $\vee$ and $\wedge$ or $<$ and $>$. This is called *the principle of duality w.r.t. order*.

The relation "$X < Y$" means X is smaller than Y and the symbol "$Y > X$" means Y is larger than X.

A Moore family is a part $\mathscr{P}$ of a complete lattice $\mathscr{P}$, where
(1) the universal element belongs to $\mathscr{P}$: $U \in \mathscr{P}$,
(2) for all non-empty parts $\mathscr{A} \subset \mathscr{P}$ we have

$$\wedge \{B : B \in \mathscr{A}\} \mid \in \mathscr{P}$$

If $\mathscr{P}_0$ is a family in $\mathscr{P}$ then the class closed under $\wedge$, generated by $\mathscr{P}_0$, united with the universal element U, constitutes a Moore family. Following our previous reasoning, a Moore family is a complete lattice. Given a family $\mathscr{P}$ of this type, there exists one and only one extensive, increasing, idempotent mapping ϕ whose invariant sets coincide with $\mathscr{P}$. This mapping is called an

algebraic closing or Moore closing, and ϕ and $\mathcal{B}$ are equivalent notions. In the theory of morphological filtering we make great use of this operation (see Chapter 5 in particular) and its dual, the algebraic opening; in the latter, the family of invariant sets is closed under sup and contains the null element.

Substituting an algebraic point of view, we can present a lattice as a set $\mathcal{P}$ equipped with two laws of composition $\vee$ and $\wedge$. Each law is

commutative: $X \wedge Y = Y \wedge X$,
associative: $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,

and they are linked by the law of absorption:

$$X \wedge (X \vee Y) = X; \qquad X \vee (X \wedge Y) = X.$$

These three axioms are the equivalent of a lattice associated with an ordering. If we restrict ourselves to the inf operation (for example) and we replace the last axiom by idempotence, i.e. $X \wedge X = X$, then the axiomatic system is equivalent to an inf semilattice.

Finally, we define a *chain* as any completely ordered subset in a lattice. If ψ is an increasing mapping of lattice $\mathcal{P}$ into itself then the image by ψ of any chain in $\mathcal{P}$ is still a chain.

This summary will suffice for the following chapter; we shall find at the beginning of Chapter 2 several complementary notions concerning distributivity and complementation. For the rest of this chapter $\mathcal{P}$ will always denote a complete lattice (see also James and James, 1976).

Examples of non-Boolean lattices

One may consider this algebraic structure to be rather far from practical applications. In fact, this structure models the *most common* procedures in applied morphology. We can see this in three examples: real-valued functions ('grey-tone morphology'), which are constantly used in morphology; partitions, less frequent, but used in segmentation problems; and topologically open sets.

Throughout this work we shall return to these three examples (Sections 1.7, 2.1 and 3.8 and Chapters 5 and 6). The confrontation between the first two of these examples is instructive. Although the first lattice is rather intuitive, the second is much less so (in particular in sup). The first is distributive, the second is not even modular (cf. Section 2.1). Monotonic continuity is better expressed in the first than in the second. Finally, the first is situated at the level of pixels, the second is on the larger scale of classes of pixels.

(a) *Lattices of u.s.c. functions* Let us consider the space of upper-semi-continuous (u.s.c.) numerical functions, bounded or not, with values in $\mathbb{R}$. We shall associate with each function $f(x)$ of this class its *umbra* $U(f)$, i.e. the set of points $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$f(x) = \sup \{t : (x, t) \in U(f)\}.$$

The ordering relation $U(f) \subset U(g)$ is equivalent to

$$\forall x : f(x) \leq g(x).$$

Conversely, if $\mathcal{U}$ denotes the class of closed umbrae, i.e.

$$\mathcal{U} = \{U : U \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}), \forall (x, t) \in U : t' \leq t \Rightarrow (x, t') \in U\},$$

then to each element $U \in \mathcal{U}$ there corresponds one and only one u.s.c. function $f(x)$ for which U is the umbra (see Chapter 9). Since the class $\mathcal{U}$ is closed for intersection and for *finite* union, it constitutes a lattice for inclusion (Fig. 1.1). If we interpret this in terms of u.s.c. functions, then for all families (finite or not) of index i we have

$$f < g \Rightarrow U(f) \subset U(g) \Rightarrow \forall x \in \mathbb{R}^n, \quad f(x) \leq g(x),$$
$$\wedge_i f_i = \{f : U(f) = \bigcap_i U(f_i)\},$$
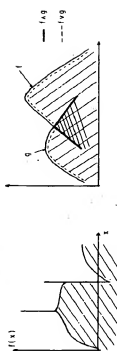$$\vee_i f_i = \{f : U(f) = \overline{\bigcup_i U(f_i)}\}.$$

Fig. 1.1 (a) Umbra of an u.s.c. function. (b) Sup and inf of two functions.

(b) *The partition lattice* Let E be an arbitrary set. We define a partition of E as the division of the set into classes such that each point $x \in E$ belongs to one and only one of these classes. More formally, we can represent a partition T as a mapping of E into $\mathcal{P}(E)$ such that

(i) $\forall x \in E, \quad x \in T(x)$,
(ii) $\forall (x, y) \in E, \quad T(x) = T(y) \quad$ or $\quad T(x) \cap T(y) = \emptyset$,

where the image $T(x)$ is the partition class that contains the point x. $\mathcal{P}$ will denote the family of partitions defined on E. We know (Simon, 1985) that we can associate the following ordering relation with $\mathcal{P}$.

$$T, T' \in \mathcal{P} : T < T' \Rightarrow T(x) \subset T'(x) \quad \forall x \in E.$$

A partition T is finer than T' when, for all x, the class $T(x)$ is included in $T'(x)$. In particular, the finest class of all has all the points of E as elements.

and the coarsest has only one element, namely the set E itself. Furthermore, this ordering allows us to construct a complete lattice. Let T_i, $i \in I$, be a family (finite or not) of partitions. We can easily see that the mapping $T: E \to \mathcal{P}(E)$, as defined by

$$T(x) = \bigcap_i T_i(x) \quad \forall x \in E,$$

generates a partition. But, by construction $T < T_i$ and for all $x \in E$, $T(x)$ is the largest element of $\mathcal{P}(E)$ that is contained in each $T_i(x)$. T is therefore the inf of T_i for ordering $<$, i.e.

$$T = \wedge_i T_i,$$

and the family $\mathcal{T}$ a complete inf semilattice with a universal element, is a complete lattice.

In this structure, the sup has a more complex expression than that of the inf. The expression $T = \vee_i T_i$ means that the partition T is the smallest partition that is larger than each T_i, or, for all x and for any i, the class $T_i(x)$ is the smallest set that is a union of classes $T_j(y)$, $y \in E$. Formally,

$$T = \bigvee_{i \in I} T_i \Leftrightarrow T(x) = \bigcup_{y \in B} T_j(y), x \in B, B \in \mathcal{P}(E)\}.$$

In partition with $\wedge T_i$, $T_i \in \mathcal{T}$ which has no particular reason to be a trivial partition, $\vee T_i$ is frequently equal to the universal element E (Fig. 1.2b).
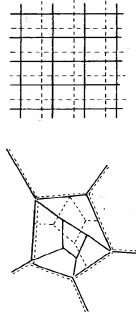

Fig. 1.2 (a) Two similar divisions of the plane into polygons. In the places where the two classes coincide, they constitute the corresponding closing class in the sup. If this is not the case, then we must continue until a "smallest common multiple" is found. (b) Two partitions (one is just the translation of the other) whose sup is R^2 itself.

(c) Lattices of open sets. Let E be a topological space, and $\mathcal{G}$ its family of open sets. We can associate the following sup and inf with set inclusion:

$$X = \bigcup \{X_i, i \in I\}, \quad X_i \in \mathcal{G},$$
$$Y = \overset{\circ}{\bigcap} \{X_i, i \in I\}, \quad X_i \in \mathcal{G}$$

The set $\mathcal{G}$ is therefore a complete lattice (by duality for complementation we see that $\mathcal{G}'$, the set of closed sets of E, is also a complete lattice).

Note that in the three examples (a), (b) and (c) there are no elements of the lattices that could be defined as points.

1.2 DILATION AND EROSION IN A COMPLETE LATTICE

We are going to study the class $\mathcal{S}(\mathcal{P})$, or more briefly $\mathcal{S}$, of mappings of $\mathcal{P}$ into itself that commute with the sup operation, such that

(1.1) $$\Gamma(\vee X_i) = \vee \Gamma(X_i), \quad i \in I, \quad X_i \in \mathcal{P},$$

and in particular $\Gamma(\emptyset) = \emptyset$. Following the notation already introduced in Euclidean mathematical morphology, we shall call this type of operation *dilation* (see Serra, 1982a). The structure of relation (1.1) shows us that dilation is an increasing operation. Moreover, any mapping of class $\mathcal{S}$ is accompanied by an equivalent transformation, called erosion, and characterized by the following theorem

Theorem 1.1 *An increasing mapping* Γ *belongs to the class* $\mathcal{S}$ *if we can associate with it another mapping* $\Gamma^*: \mathcal{P} \to \mathcal{P}$ *such that*

(1.2) $$\Gamma(X) < Y \Leftrightarrow X < \Gamma^*(Y), \quad X, Y \in \mathcal{P}.$$

The mapping Γ^* *is thus unique and increasing, and is given by*

(1.3) $$\Gamma^*(X) = \vee \{B: B \in \mathcal{P}, \Gamma(B) < X\}.$$

Proof We shall first show that (1.2) $\Rightarrow$ (1.3) by considering the mapping

$$\Gamma^*(X) = \vee \{B: B \in \mathcal{P}, \Gamma(B) < X\}.$$

$\Gamma(B) < X \Rightarrow \Gamma^*(X) < \Gamma^*(X)$; thus $\Gamma^*\Gamma^*(X) < \Gamma^*(X)$. Conversely $\Gamma^*(X) < \Gamma^*(X)$, thus by (1.2) $\Gamma \circ \Gamma^*(X) < X$ and $\Gamma(X) < \Gamma \circ \Gamma^*(X)$. This gives us $\Gamma^* = \hat{\Gamma}$ and proves uniqueness, since if Γ exists it must be of the form (1.3).

Now let us look at (1.2) $\Rightarrow$ (1.1). Let $\{X_i\}$ be a family of elements of $\mathcal{P}$. Applying (1.2) by taking X for X and $\vee \Gamma(X_i)$ for Y, we have

$$\Gamma(X_i) < \vee_j \Gamma(X_j) \Rightarrow \forall i, X_i < \Gamma^*(\vee \Gamma(X_i)) \Rightarrow \vee X_i < \Gamma^*(\vee \Gamma(X_i))$$
$$\Leftrightarrow \Gamma(\vee X_i) < \vee \Gamma(X_i).$$

We also find the inverse inclusion, and, since Γ is increasing, (1.1) follows.

Conversely (1.1) $\Rightarrow$ (1.2). To any mapping $\Gamma \in \mathcal{S}$ we can associate its erosion $\hat{\Gamma}$, as defined by (1.3). The form of (1.3) shows that $\Gamma(B) < X \Leftrightarrow B < \hat{\Gamma}(X)$.

For the inverse implication, since Γ commutes with sup, we have $B < \dot{\Gamma}(X)$ $\Rightarrow \Gamma(B) < \dot{\Gamma}\Gamma(X) \Rightarrow B \in \mathscr{P}, \Gamma(B) < X \} < X$. Finally, the growth of $\dot{\Gamma}$ can be seen directly in relation (1.3). ∎

Remark Since the only operation occurring explicitly in the above discussion was sup, one might think that the theorem uses only the axioms of semilattices. In fact the definition of erosion, (1.3), assumes the existence of the null element in $\mathscr{P}$ (X may not contain a $\Gamma(B)$); then $\mathscr{P}$ is a complete semilattice with null and universal elements ($\emptyset$ and U) and is thus a complete lattice with inf

$$\inf\{\Gamma_i : \Gamma_i \in \mathscr{S}\} = V\{M : M \in \mathscr{M}\} = m.$$

In contrast with the sups in $\mathscr{P}$ and $\mathscr{S}$, which are directly linked, the infs are not linked at all.

(f) Dilation and erosion are increasing mappings. The derivation of the former follows directly from the form of (1.1), since

$$X_1 < X_2 \Rightarrow \Gamma(X_1) < \Gamma(X_2), \quad \dot{\Gamma}(X_1) < \dot{\Gamma}(X_2).$$

As to the latter, we deduce it from (1.3), since

Properties of dilation and erosion

(a) Erosion commutes with the operation inf. Indeed, since $\dot{\Gamma}$ is increasing, we have $\dot{\Gamma}(\wedge X_i) < \wedge \dot{\Gamma}(X_i)$, but also

$$X_i > \Gamma(\wedge \dot{\Gamma}(X_i)) \Rightarrow \dot{\Gamma}(X_i) = X_i > \Gamma(\wedge \dot{\Gamma}(X_i))$$
$$\Rightarrow \dot{\Gamma}(\wedge X_i) > \dot{\Gamma}(X_i).$$

(b) $\dot{\Gamma}(U) = U$, because $\Gamma(U) > U \Rightarrow U \subset \dot{\Gamma}(U)$.

(c) The class $\mathscr{S}$ constitutes a semigroup and contains a neutral element, the identity I, since for each pair $(\Gamma_1 \Gamma_2) \in \mathscr{S}$

$$(\Gamma_1 \Gamma_2)(\vee X_i) = \Gamma_1 \Gamma_2(\vee \Gamma_i(X_i)) = \vee \Gamma_1 \Gamma_2(X_i).$$

Iterating, we see that $\Gamma_1 \dot{\Gamma}(\Gamma_2) = (\Gamma_1 \Gamma_2)\dot{\Gamma}_1$. We then deduce that erosions also form a semigroup since

$$(\Gamma_1 \Gamma_2)(\vee X_i) = \vee \{B : B \in \mathscr{P}, (\Gamma_1 \Gamma_2)(B) < X\}$$
$$= \vee \{B : B \in \mathscr{P}, \Gamma_1(B) < \dot{\Gamma}_2(X)\} = \dot{\Gamma}_2 \dot{\Gamma}_1(X).$$

(d) If Γ_1 and $\dot{\Gamma}_1$ are ordered then so are Γ_i and $\dot{\Gamma}_i$, but in the opposite sense. Thus

$$[\forall X : \Gamma_1(X) < \Gamma_2(X)] = [\forall X : B < \Gamma_1(X) \Rightarrow B < \dot{\Gamma}_2(X)]$$
$$= [\forall X : \dot{\Gamma}_1(X) < \dot{\Gamma}_2(X)].$$

(e) The class $\mathscr{S}$ has the structure of a complete lattice. If suffices to equip $\mathscr{S}$ with an ordering relation

$$\Gamma_1 < \Gamma_2 \Rightarrow \Gamma_1(X) < \Gamma_2(X) \quad \forall X \in \mathscr{P}$$
$$\text{in } \mathscr{S} \qquad \qquad \text{in } \mathscr{P}$$

We then find the equality $(\vee \Gamma_i)(X) = \vee \Gamma_i(X)$, whose second member belongs to $\mathscr{S}$, since

$$\vee \Gamma_i(\vee X_j) = \vee_i \vee_j \Gamma_i(X_j) = \vee_j \vee_i \Gamma_i(X_j).$$

$\mathscr{S}$ is therefore a sup-semilattice; now $\mathscr{S}$ has a least element $\Gamma_\emptyset(X) = \emptyset$, $\forall X \in \mathscr{P}$. Consequently, if Γ_i is a family in $\mathscr{S}$ then the set of lower bounds $\mathscr{M}$ is not empty since it contains $\Gamma_\emptyset$. As $\mathscr{S}$ is a complete sup-semilattice, $\mathscr{M}$ must have a greatest element m, which is the greatest lower bound of the Γ_i. $\mathscr{S}$ is therefore a complete lattice.

We are going to show that this property has a converse and that any increasing mapping ψ of $\mathscr{P}$ into itself such that $\psi(U) = U$ can be represented as the sup of erosions. Let $\psi : \mathscr{P} \to \mathscr{P}$ be an increasing mapping. Given

$$\Gamma_\psi(Y) = \begin{cases} \emptyset & \text{if } Y = \emptyset, \\ B & \text{if } Y < \psi(B), \\ U & \text{if } Y \text{ is not included in } \psi(B). \end{cases}$$

$\Gamma_\psi(Y)$ satisfies relation (1.1). It is therefore a dilation. To determine the corresponding erosion, we must first distinguish the case where $X > B$ from the contrary in the expression

$$(1.4) \qquad \dot{\Gamma}_\psi(X) = \vee\{Y : Y \in \mathscr{P}, \Gamma_\psi(Y) < X\}.$$

As with $\Gamma_\psi(X)$, the erosion $\dot{\Gamma}_\psi(X)$ can have only three values. Obviously, if $X = U$ then $\dot{\Gamma}_\psi(U) = U$. If $X \neq U$ and $X > B$ then the inequality $\Gamma_\psi(Y) < X$ is satisfied if and only if $Y < \psi(B)$, since the sup of these Y is $\psi(B)$, we have $\dot{\Gamma}_\psi(X) = \psi(B)$. If X is not larger than B then either $Y < \psi(B)$, giving $\Gamma_\psi(Y) < B$ and Y is not smaller than X, where $\Gamma_\psi(X)$ is defined by (1.4), or Y is not smaller than $\psi(B)$, and since $X > U$ (otherwise it would be larger than B), $\Gamma_\psi(Y) = U$ is not smaller than X. Finally, $\dot{\Gamma}(X)$ is reduced to the null element:

$$\dot{\Gamma}_\psi(X), B \in \mathscr{P} = \begin{cases} U & \text{if } X = U, \\ \psi(B) & \text{if } X > B \text{ and } X \neq U, \\ \emptyset & \text{otherwise.} \end{cases}$$

Taking the sup of the $\dot{\Gamma}_\psi(X)$ when B spans $\mathscr{P}$ and considering the growth of ψ, we have

$$\vee[\dot{\Gamma}_\psi(X), B \in \mathscr{P} : B < X] = \psi(X).$$

Conversely, since erosion is increasing, the upper bound of an arbitrary family of erosions is also increasing. We can now state the following

Theorem 1.2 *The class of mappings* $\Gamma : \mathscr{P} \to \mathscr{P}$ *of a complete lattice into itself, and that commute with sup, constitutes a complete lattice of increasing mappings. Moreover, any mapping* $\psi : \mathscr{P} \to \mathscr{P}$ *such that* $\psi(U) = U$ *is increasing iff it can take the form of an upper bound of erosions. More precisely, the mapping* ψ *can be written as*

$$\psi = \vee\{\epsilon_B, B \in \mathscr{B}\},$$

with $\Gamma_B(X)$ *if* $X > B$ *and* $\psi_B(X) = \emptyset$ *otherwise.*

1.3 MORPHOLOGICAL OPENINGS AND CLOSINGS

In general, the mappings $X \to \Gamma(X)$ and $X \to \check{\Gamma}(X)$ do not have inverses, and there is no way to determine X from its images $\Gamma(X)$ or $\check{\Gamma}(X)$. But this indeterminacy is only partial. We shall always have either an upper or a lower bound, depending on the situation at hand.

For example, consider $\check{\Gamma}(X)$. This is, as we saw in (1.3), the sup of all Bs whose images under Γ are smaller than X. Consequently, the sup of all these images constitutes the minimal inverse image of $\check{\Gamma}(X)$. In more algebraic terms: if we take in (1.2) $\check{\Gamma}(Y)$ for the set X, the inequality $\Gamma \check{\Gamma}(Y) > \check{\Gamma}(X)$ is equivalent to

$$\check{\Gamma}\Gamma(Y) < Y \quad \forall Y \in \mathscr{P}$$

In the same way, taking $Y = \Gamma(X)$ in (1.2), we find

$$X < \check{\Gamma}\Gamma(X) \quad \forall X \in \mathscr{P}$$

Using the generic symbols for openings and closings, γ and ϕ, let us write

$$\gamma_\Gamma = \Gamma\check{\Gamma} \quad \text{and} \quad \phi_\Gamma = \check{\Gamma}\Gamma.$$

Actually, γ_Γ and ϕ_Γ are idempotent. For example, $\Gamma\check{\Gamma} > \Gamma$ implies $\gamma_\Gamma\gamma_\Gamma = \Gamma\check{\Gamma}\Gamma\check{\Gamma} > \Gamma\check{\Gamma}$, but since γ_Γ is anti-extensive, we have $\gamma_\Gamma\gamma_\Gamma < \gamma_\Gamma$, and thus the desired result. To avoid confusion concerning openings and closings and the more general algebraic concepts attached to these terms (Section 1.1), we shall call them *morphological* (and remove the index Γ where there is no ambiguity). Explicitly,

$$(1.5) \quad \begin{cases} \gamma(Y) = \gamma_\Gamma(X) = \vee\{\Gamma B, B \in \mathscr{B} : \Gamma(B) < X\}, \\ \phi(X) = \phi_\Gamma(X) = \vee\{B, B \in \mathscr{B} : \Gamma(B) < \Gamma(X)\}. \end{cases}$$

We know that algebraic openings (resp. closings) are characterized by their invariant domains (cf. Sections 1.1 and 5.4). In the present case it is just the

form of the defining algorithm, $\gamma = \Gamma\check{\Gamma}$, that shows that if B is open by γ, i.e. if $\gamma(B) = B$, then B is of the type $B = \Gamma(Z)$. Conversely, if Z is an arbitrary element of $\mathscr{P}$ then by extensivity of closing $\gamma\Gamma(Z) = \Gamma\check{\Gamma}\Gamma(Z) > \Gamma(Z)$, and since γ is an opening, we have $\gamma\Gamma(Z) < \Gamma(Z)$. Thus the family $\mathscr{A}_\gamma$ of invariant sets of the morphological opening γ is the image $\Gamma(\mathscr{P})$ of $\mathscr{P}$ under Γ. Similarly, for the closing we find $\mathscr{A}_\phi = \check{\Gamma}(\mathscr{P})$, since $Z \in \mathscr{A}_\phi$ implies $\check{\Gamma}\Gamma\check{\Gamma}(Z) < \check{\Gamma}(Z)$, because $\check{\Gamma}\Gamma$ is an opening, and also $(\check{\Gamma}\Gamma)\check{\Gamma}(Z) > \check{\Gamma}(Z)$, since $\Gamma\check{\Gamma}$ is a closing, and finally $\phi(Z) = \check{\Gamma}(Z)$. To summarize, we have the following

Theorem 1.3 *The products* $\gamma = \Gamma\check{\Gamma}$ *and* $\phi = \check{\Gamma}\Gamma$ *define respectively morphological opening and closing on the lattice* $\mathscr{P}$. *The class of invariant sets of the former is the image of* $\mathscr{P}$ *under* Γ, *and that of the latter forms the image of* $\mathscr{P}$ *under* $\check{\Gamma}$.

Algebraic openings and morphological openings

We can easily see that any upper bound of morphological openings γ_i, with corresponding domains of invariance $\mathscr{A}_i$, is yet another opening, but in this case not morphological. It has as its domain of invariance the class closed under the sup (in $\mathscr{P}$) generated by the union of the $\mathscr{A}_i$ (in the space $\mathscr{P}(\mathscr{P})$ of the subsets of $\mathscr{P}$). Moreover, this property is valid for any opening, and is a classical version of the theory of morphological filtering (see Section 5.4).

Now consider the converse problem. Starting with an arbitrary opening $\gamma : \mathscr{P} \to \mathscr{P}$ we can consider it as a sup of morphological openings? We know from Matheron (1975) and Proposition 5.3 of Chapter 5 that any algebraic opening γ on $\mathscr{P}$ with invariant domain $\mathscr{A}_\gamma$ is the smallest extension to $\mathscr{P}$ of the identity on $\mathscr{A}_\gamma$, and is written

$$(1.6) \quad \gamma(X) = \vee\{B : B \in \mathscr{A}_\gamma, B < X\} \quad \forall X \in \mathscr{P}$$

The class $\mathscr{A}_\gamma$ is closed under sup. Conversely, the class closed under union generated by an arbitrary class, $\mathscr{A}_i \in \mathscr{P}(\mathscr{P})$, defines, with the aid of (1.6), a mapping that is an opening. Now, associate with each $B \in \mathscr{B}$ the dilation

$$\Gamma_B(A) = B \quad \text{if } A \text{ not} < B; \qquad \Gamma_B(A) = \emptyset \quad \text{if } A < B.$$

Its corresponding morphological opening is

$$\gamma_B(X) = \vee\{\Gamma_B(A) : \Gamma_B(A) < X\} = \begin{cases} B & \text{if } B < X, \\ \emptyset & \text{if } B \text{ not} < X, \end{cases}$$

and (1.6) is consequently equivalent to

$$\gamma = \vee\{\gamma_B, B \in \mathscr{B}\}.$$

In other words, we have the following.

Theorem 1.4 *If the mapping* $\psi : \mathscr{P} \to \mathscr{P}$ *is an opening then it has a representation of the form*

$$(1.6) \qquad \psi = \vee\{\gamma_B, B \in \mathscr{B}\}_i$$

where γ_B *is the morphological opening associated with the dilation* $\Gamma_B(A) = \emptyset$ *if* $A < B$ *and* $\Gamma_B(A) = B$ *otherwise, and where* $\mathscr{B}$ *is the domain of invariance of* ψ. *Conversely, if* $\mathscr{B}$ *denotes the class closed under union generated by* $\mathscr{B}_i$, *a class of arbitrary elements of* $\mathscr{P}$, *then the mapping defined by (1.6) is an opening.*

Size distribution

Matheron (1975) proposed an axiomatic definition of what one intuitively understands by size distribution (see Section 5.5). His definition brings into play families of (algebraic) openings, γ_λ, depending on a positive parameter λ such that

$$(1.7) \qquad \lambda \geq \mu \Rightarrow \gamma_\lambda = \gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda.$$

We propose to characterize the family of dilations Γ_λ, whose openings can be used to generate size distributions. With this aim, consider the class of Γ_λ such that

$$(1.8) \qquad \lambda \geq \mu > 0 \Rightarrow \Gamma_\mu = \gamma_\mu \Gamma_\lambda.$$

To each element $Y \in \mathscr{P}$, associate its erosion $X = \Gamma_\lambda(Y)$ and apply the algorithm (1.8). Thus

$$\Gamma_\lambda \gamma_\lambda(Y) = \gamma_\lambda \Gamma_\lambda(Y), \qquad \text{i.e. } \gamma_\lambda = \gamma_\lambda \gamma_\lambda.$$

Moreover, idempotence implies $\gamma_\lambda < \gamma_\lambda \gamma_\lambda \gamma_\lambda < \gamma_\lambda \gamma_\lambda$, and we have, by growth, $\gamma_\lambda > \gamma_\lambda \gamma_\lambda$. So (1.8) implies (1.7).

Conversely, apply the first equality from (1.7) to the element $\Gamma_\lambda(X)$, $X \in \mathscr{P}$. We find $\Gamma_\lambda \Gamma_\lambda \Gamma_\lambda = \gamma_\lambda \Gamma_\lambda \Gamma_\lambda$. But on the one hand, we have

$$\Gamma_\lambda < \Gamma_\lambda(\Gamma_\lambda \Gamma_\lambda) = \gamma_\lambda \Gamma_\lambda \Gamma_\lambda \Gamma_\lambda < \gamma_\lambda \Gamma_\lambda,$$

and, on the other, we also have

$$\Gamma_\lambda > (\Gamma_\lambda \Gamma_\lambda)\Gamma_\lambda = \gamma_\lambda \Gamma_\lambda(\Gamma_\lambda \Gamma_\lambda) > \gamma_\lambda \Gamma_\lambda.$$

and (1.7) implies (1.8), which characterizes the classes Γ_λ that can generate size distributions. We notice that no ordering relation of the Γ_λ was required.

In summary, we have the following.

Theorem 1.5 *The morphological openings associated with any family* $\{\Gamma_\lambda\}$ *of dilations depending on a positive parameter* λ *generate a size distribution over the lattice* $\mathscr{P}$ *iff for any* λ, Γ_λ *is morphologically open by the* γ_λ, *where* $\mu \leq \lambda$, *that is to say,* $\lambda \geq \mu$ *implies* $\Gamma_\lambda = \gamma_\lambda \Gamma_\lambda$.

Remarks

(i) We can easily see that the axiom (1.7) is equivalent to ordering the (morphological or algebraic) openings by λ:

$$(1.7) \Rightarrow \{\lambda \geq \mu > 0 \Rightarrow \gamma_\lambda < \gamma_\mu\}.$$

(ii) In the same manner, we have introduced size distributions from their invariant sets. We shall see in Section 5.5 that the axiom (1.7) is also equivalent to

$$\lambda \geq \mu > 0 \Rightarrow \mathscr{B}(\gamma_\lambda) \subset \mathscr{B}(\gamma_\mu).$$

(iii) Taking the dual point of view, we similarly define antisize distributions as being families of algebraic closings $\{\phi_\lambda\}$, depending on the positive parameter λ, such that one of the following three equivalent axioms is satisfied:

$$\lambda \geq \mu > 0 \Rightarrow \phi_\lambda \phi_\mu = \phi_\mu \phi_\lambda = \phi_\lambda,$$
$$\lambda \geq \mu > 0 \Rightarrow \phi_\lambda > \phi_\mu,$$
$$\lambda \geq \mu > 0 \Rightarrow \mathscr{B}(\phi_\lambda) \subset \mathscr{B}(\phi_\mu).$$

(iv) As an example of sizing families, $\{\Gamma_\lambda\}$, we can take semigroups of type $\Gamma_\lambda = \Gamma_1^n \Gamma_{(\mu+\nu)}$, $\lambda \geq \mu \geq 0$, which will often be used in the sequel. We can elaborate discrete versions of them, which are nonetheless size distributions, from iterations of dilations Γ^n of order k, since by associativity $\Gamma^n = \Gamma^{n-k}\dots$ ($n - k$ positive integers) (see Section 4.3).

1.4 DUAL ASPECT, THE POINT OF VIEW OF EROSIONS

If until now we have given priority to the concept of dilation, it was for pedagogical reasons. An initial set of results without too much consideration of the fact that each one has its analogue based on erosion. It goes without saying that the symmetry of the roles of sup and inf in lattice $\mathscr{P}$ hold for all the logical constructions built upon it, and this leads us to looking at the few preceding theorems from the view point of duality. Consider the first of these, (1.1). To do this, consider the class of operations called erosions, $\mathscr{S}^*$, which commute with inf, i.e. such that

$$(1.9) \qquad \Gamma^*(\wedge_i X_i) = \wedge_i \Gamma^*(X_i), \qquad i \in I, \quad X_i \in \mathscr{P}$$

and such that $\Gamma^*(U) = U$. Saying that Γ^* belongs to $\mathscr{S}^*$ is equivalent to saying that there exists a dilation Γ, $\Gamma \in \mathscr{S}$ such that

$$(1.2) \qquad \Gamma_*(X) < Y \Leftrightarrow X < \Gamma^*(Y) \quad \forall \text{ pairs } (X, Y) \in \mathscr{P}$$

This dilation is unique, and is given by

(1.10) $\Gamma(X) = \wedge\{B : B \in \mathcal{P}, \Gamma^*(B) > X\}$.

Conversely, if we start with an arbitrary dilation Γ, the relation (1.2) produces a corresponding erosion Γ^* that is unique and commutes with inf. The preceding discussion tells us that we can use the algorithm (1.10) to determine the associated dilation Γ''. But since

$\Gamma''(X) = \wedge\{B : B \in \mathcal{P}, X < \Gamma(B)\} = \wedge\{B, B \in \mathcal{P}, \Gamma(X) < B\} = \Gamma(X)$,

it follows that Γ^* and Γ'' coincide. There is an isomorphism between $\mathcal{P}$ and $\mathcal{P}$, and the latter is also a complete lattice. From this we obtain the following.

Theorem 1.6 *Let $\mathcal{P}$ be a complete lattice, $\mathcal{S}$ the class of mappings $\Gamma: \mathcal{P} \to \mathcal{P}$ that commute with sup, and $\mathcal{S}^*$ the class of mappings Γ^* that commute with inf. $\mathcal{S}$ and $\mathcal{S}^*$ are two complete isomorphic lattices, which correspond to one another through the duality relation*

$\Gamma(X) < Y \Leftrightarrow X < \Gamma^*(Y)$ $\forall X \in \mathcal{P}, Y \in \mathcal{P}$.

To each dilation $\Gamma \in \mathcal{S}$ there corresponds an erosion $\Gamma^* \in \mathcal{S}^*$:

$\Gamma^*(X) = \vee\{B : B \in \mathcal{P}, \Gamma(B) < X\}$.

and to each erosion $\Gamma^* \in \mathcal{S}^*$ there corresponds a dilation $\Gamma \in \mathcal{S}$

$\Gamma(X) = \wedge\{B : B \in \mathcal{P}, \Gamma^*(B) > X\}$.

Remarks

(i) This theorem suggests a tautological expression of Γ and Γ^*. For example, each Γ can be written as

$\Gamma(X) = \wedge\{B : B \in \mathcal{P}, \vee\{C, C \in \mathcal{P}, \Gamma(C) > B\} > X\}$.

(ii) Theorem (1.2) is easily transposed, and any increasing mapping ψ such that $\psi(\emptyset) = \emptyset$ can be interpreted as an inf of dilations. In the same manner, any algebraic closing can be interpreted as an inf of morphological closings (Theorem 1.4), and any family of erosions $\{\Gamma^*_i\}$ defines an antisize distribution ϕ_λ if $\lambda \geq \mu$ implies $\Gamma^* = \phi_\mu \Gamma^*_\lambda$ (Theorem 1.5). Theorem 1.3 remains unchanged.

1.5 MONOTONE CONTINUITY IN $\mathcal{P}$ AND $\mathcal{S}$

If we wish to remain in the general algebraic framework of the lattice $\mathcal{P}$, and nevertheless wish to describe operations such as size distributions, then a

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certain amount of continuity is very useful. All that is available to us is *monotone continuity*. We can express this at two levels of generality. Given an increasing mapping, $\psi: \mathcal{P} \to \mathcal{P}$, then either, for each increasing sequence $\{X_i\}$, n an integer, we have

$\psi(\vee X_i) = \vee\psi(X_i)$

(which is written $X_i \uparrow X \Rightarrow \psi(X_i) \uparrow \psi(X)$), and we are then working with *sequential* monotone continuity; or, for each filtering family $\{X_i\}, i \in I$, where I is an ordered set and $i < j \Rightarrow X_i < X_j$, we have

$\psi(\vee X_i, i \in I) = \vee\psi(X_i)$,

and we are then dealing with *general* monotone continuity. The latter is much stronger than the former (note once again that $X_i \uparrow X = \psi(X) \uparrow \psi(X)$).

Obviously, by duality these notions extend to cover decreasing sequences as well as decreasing mappings. Between these two notions... the sequential case where n spans the set of positive integers $\mathbb{Z}^+$, and the general case of the ordered space I, we find the case where the index I of X describes a subset of $\mathbb{R}$. However, this case is equivalent to sequential monotone continuity. Indeed, if the family $\{\lambda\}$ is bounded above by λ_0, and X_λ increases w.r.t. λ, then

$X_{\lambda_0} = \vee\{X_\lambda : \lambda < \lambda_0\}$ is equivalent to $X_{\lambda_0} = \vee X_{\lambda_{n-1}}$.

If the increasing mapping ψ satisfies sequential monotone continuity, i.e.

$\psi(X_{\lambda_0}) = \psi(\vee X_{\lambda_{n-1}}) = \vee\psi(X_{\lambda_{n-1}})$,

then it follows that $\psi(X_{\lambda_0}) = \vee\{\psi(X_\lambda) : \lambda < \lambda_0\}$ and sequential monotone continuity is sufficient. As it is obviously necessary (evident if we begin with $X_\lambda = \vee\{X_\lambda : n < \lambda\}$), it is *equivalent* to the monotone continuity in λ on $\mathbb{R}^2$.

In the following pages we shall rarely use general monotone continuity, and we shall always specify it explicitly. Otherwise, we shall speak of $\downarrow$ continuity or $\uparrow$ continuity, and similarly we shall say that a mapping is *continuous* when it is both $\uparrow$ and $\downarrow$ continuous.

Relation (1.1), which defines the class of dilations, implies their $\uparrow$ continuity in the general sense, and, by duality, the $\downarrow$ continuity of erosions. It follows from this that if Γ is $\downarrow$ continuous (and thus continuous) then $\gamma = \Gamma\Gamma^*$ and $\phi = \Gamma^*\Gamma$ are also $\downarrow$ continuous, since Γ^* is always $\downarrow$ continuous, by duality γ and ϕ are $\uparrow$ continuous when Γ is.

However, $\uparrow$ continuity for Γ does not imply $\uparrow$ continuity for Γ^*. In general, neither openings nor closings are continuous. In summary, we have the following.

Theorem 1.7 *Dilation is a $\uparrow$ continuous mapping of $\mathcal{P}$ into itself and erosion is a $\downarrow$ continuous mapping. Beyond that, if dilation Γ is continuous*

then opening $\Gamma\Gamma'$ and closing $\Gamma'\Gamma$ are $\uparrow$ continuous. In the same manner, if Γ' is continuous then $\Gamma\Gamma'$ and $\Gamma'\Gamma$ are $\downarrow$ continuous. All of these continuities must be understood in the sense of general monotone continuity.

Remark The use here of the term "continuous" is consistent with its topological sense. In fact, any chain in a lattice $\mathcal{P}$ (i.e. any completely ordered subset) constitutes a topological space for the ordering topology. The increasing mappings on $\mathcal{P}$, which transform a chain into a chain, therefore connect two topological spaces. We thus introduce a topology, but only if we restrict ourselves to monotone sequences. We shall see that for a great many problems this constraint is largely acceptable. In exchange, we shall be able to define concepts that would otherwise have been inaccessible. One example of this is the alternating sequential filter in Chapter 10 (this remark also applies to left- and right-continuities).

Left- and right-continuity

In the preceding discussion we assumed X to be variable and Γ fixed. However, certain concepts, such as size distribution, bring us to families of dilations $\{\Gamma_\lambda\}$, which depend on a scalar λ.

We shall say that the mapping $\lambda \to \Gamma$ from R^* into $\mathcal{P}$ is increasing if for all $X \in \mathcal{P}$ we have

$$\lambda \geq \lambda' \to \Gamma_\lambda(X) > \Gamma_{\lambda'}(X)$$

and that it is decreasing if for all X

$$\lambda \geq \lambda' \to \Gamma_\lambda(X) < \Gamma_{\lambda'}(X).$$

If the mapping $\lambda \to \Gamma_\lambda$ is increasing, we say $\Gamma_\lambda \downarrow \Gamma_{\lambda_0}$ if and only if $\lambda \downarrow \lambda_0$ implies $\Gamma_\lambda(X) \downarrow \Gamma_{\lambda_0}(X)$ for all $X \in \mathcal{P}$ (resp. $\Gamma_\lambda \uparrow \Gamma_{\lambda_0}$ when $\lambda \uparrow \lambda_0$). By duality, this property transfers to decreasing mappings $\lambda \to \Gamma_\lambda$.

To avoid confusion between the convergence of $\Gamma_\lambda(X)$ with X variable or with λ variable, we apply the term left-continuity (or right-continuity) to mappings, $\lambda \to \Gamma_\lambda$, $\lambda \to \Gamma_\lambda$, etc. Specifically, Γ is left-continuous when $\lambda \uparrow \lambda_0$ implies $\Gamma_\lambda \uparrow \Gamma_{\lambda_0}$ (or $\Gamma_\lambda \downarrow \Gamma_{\lambda_0}$). We obtain right-continuity by starting at $\lambda \downarrow \lambda_0$. Finally, for mappings such as $(\lambda, X) \to \Gamma_\lambda(X)$ or $(\lambda, X) \to \gamma_\lambda(X)$, which combine variations in X and λ, we shall use the general terminology $\uparrow$ and $\downarrow$ continuity.

If the mapping $\lambda \to \Gamma_\lambda$ is increasing and left-continuous then its dual $\to \hat{\Gamma}_\lambda$ is decreasing and right-continuous. We showed it to be decreasing in Section 1.2, (b) on p. 18. To establish left-continuity, start with the following equivalences for all pairs $X, Y \in \mathcal{P}$:

$$\Gamma_{\lambda_0} = \vee[\Gamma_\lambda(X) < Y : \lambda < \lambda_0] \Rightarrow \forall \lambda < \lambda_0,\ \Gamma_\lambda(X) < Y$$
$$\Rightarrow \forall \lambda < \lambda_0,\ X < \hat{\Gamma}_\lambda(Y)$$
$$\Rightarrow X < \wedge[\hat{\Gamma}_\lambda(Y): \lambda < \lambda_0].$$

Thus

$$\hat{\Gamma}_{\lambda_0}(Y) = \wedge[\hat{\Gamma}_\lambda(X): \lambda < \lambda_0] \qquad \blacksquare$$

Note that in general, a right-limit of dilations (as defined by an inf) need not be a dilation.

Semicontinuity and size distributions

Proposition 1.8   *If the dilation $\Gamma_\lambda: \mathcal{P} \to \mathcal{P}$ is continuous for all λ and the mapping $\lambda \to \Gamma_\lambda$ is increasing and left-continuous then the size distribution $(\lambda, X) \to \gamma_\lambda(X)$ that maps $R^* \times \mathcal{P}$ into $\mathcal{P}$ is $\downarrow$ continuous.*

Proof

Given two sequences, $\lambda \uparrow \lambda_0$ and $X_i \uparrow X_\infty$ and setting $\Gamma_\lambda = X_\infty$, we have not only

$$\gamma_{\lambda_0}(X_\infty) = \hat{\Gamma}_{\lambda_0}[\Gamma_{\lambda_0}(X_\infty): \lambda < \lambda_0]$$
$$= \wedge[\hat{\Gamma}_\lambda[\Gamma_\lambda(X_\infty): \lambda < \lambda_0] > \wedge[\gamma_\lambda(X_\infty): \lambda < \lambda_0]$$

but also $\gamma_{\lambda_0} < \gamma_i$ for all $\lambda < \lambda_0$. Therefore $\gamma_{\lambda_\infty} < \wedge_{\lambda < \lambda_0} \gamma_i$, which gives us the equality. Since Γ_λ is continuous relative to X, we can write

$$\gamma_{\lambda_0}(X_\infty) = \wedge[\gamma_{\lambda_0}(X_p): \lambda < \lambda_0] = \wedge[\gamma_{\lambda_0}(X_p): \rho \geq_i, \lambda < \lambda_0].$$

Remarks

(i)   We could have obtained a similar result by assuming $\lambda \to \Gamma_\lambda$ to be right-continuous. In this case it would not have been necessary to take the hypothesis of $\downarrow$ continuity for $X \to \Gamma_\lambda(X)$. The two conditions ($\lambda \uparrow \lambda_0$, $\Gamma_\lambda \downarrow \lambda \lambda_0$, (eventually $X \lambda_0$) = 0) and $X \downarrow X_\infty$ would then imply $\gamma \lambda_0(X_\lambda)$. These two versions foreshadow the semicontinuous behaviour that we meet when dealing with spaces of closed and open sets $\mathcal{F}$ and $\mathcal{G}$ of a locally compact, separable, Hausdorff space (denoted below by LCS; cf. for examples Matheron (1975, p. 25) or (Serra, 1982a, Chapter III).

(ii)   As to invariant sets, left-continuity of Γ_λ is equivalent to the implication $\lambda \uparrow \lambda_0 \Rightarrow \mathcal{G}(\gamma_\lambda) \downarrow \mathcal{G}(\gamma_{\lambda_0})$.

(iii)   We find the dual results for antisize distributions by applying the same hypothesis while exchanging "left" for "right" and reversing the arrows.

Extreme elements

When describing an element of $\mathcal{P}$ by studying its evolution under the action of a family of decreasing erosions or openings, it is interesting to look at the last step in this evolution: that is to say, when the set disappears. Mathematical morphology has introduced two related concepts to handle this: the ultimate erosion, for erosions, and the critical element, for openings.

(a) *Ultimate erosions* Suppose we have a family $\{\Gamma^\lambda_\mu\}$ that is decreasing w.r.t a positive λ and completed by $\Gamma^\lambda_0 = I$. Let λ vary over the interval $[0, \lambda_{max}]$ and consider $\mathcal{P}(\lambda_{max})$, the class of elements $X \in \mathcal{P}$ whose erosion $\Gamma_{\lambda_{max}}(X)$ is empty. As $\Gamma^\lambda_0 = I$ and Γ^λ_λ decreases as λ increases, there exists a value λ_0 for each $X \in \mathcal{P}(\lambda_{max})$ such that

(1.11) $\lambda < \lambda_0 \Rightarrow \Gamma_\lambda(X) \neq \emptyset, \qquad \lambda > \lambda_0 \Rightarrow \Gamma_\lambda(X) = \emptyset$

or

$$\lambda_0(X) = \sup\{\lambda : \Gamma_\lambda(X) \neq \emptyset\}, \qquad X \in \mathcal{P}(\lambda_{max}).$$

We then call $\Gamma^\lambda_{\lambda_0}(X)$ the *ultimate erosion* of X w.r.t. the family $\{\Gamma_\lambda\}$. When the mapping $\lambda \to \Gamma$ is left-continuous, this element is determined by the erosions of lower index, since

$$\Gamma_{\lambda_0}(X) = \wedge\{\Gamma_\lambda(X); \lambda < \lambda_0\}.$$

(b) *Critical elements of a size distribution* As above, we can associate with each size distribution its critical elements (Matheron 1975, p. 194). The set X is critical for $\lambda = \lambda_0$ when

$$\lambda < \lambda_0 \Rightarrow \gamma_\lambda(X) \neq \emptyset; \qquad \lambda > \lambda_0 \Rightarrow \gamma_\lambda(X) = \emptyset$$

and $\lambda_0 = \sup\{\lambda : \gamma_\lambda(X) \neq \emptyset\}$. When $\lambda \uparrow \lambda_0$ implies $\gamma_\lambda \downarrow \gamma_{\lambda_0}$ we have

$$\gamma_{\lambda_0}(X) = \wedge\{\gamma_\lambda(X); \lambda < \lambda_0\} = M,$$

and, by idempotence for opening, $\gamma_\lambda(M) = M$. Since we know that $\mathcal{B}_{\lambda_0} \subset \mathcal{B}_\lambda$ for $\lambda \leq \lambda_0$, M is invariant for all openings smaller than λ_0:

$$\lambda \leq \lambda_0 \Rightarrow \gamma_\lambda(M) = M, \qquad \lambda > \lambda_0 \Rightarrow \gamma_\lambda(M) = M.$$

Conversely, to each $M \in \mathcal{P}$ such that $\gamma_{\lambda_0}(M) = \emptyset$, there corresponds a $\lambda_0 \in [0, \lambda_{max}]$ such that M is critical for λ_0.

Thus, subject to families of transformations $\{\Gamma_\lambda\}$ and $\{\gamma_\lambda\}$ that progressively reduce any $X \in \mathcal{P}$ to the null element, these families produce two new mappings of $\mathcal{P}$ into $\mathbb{R}^+$: the ultimate erosion and the critical opening. We interpret them as descriptors of the maximum sizes for the elements of $\mathcal{P}$.

1.6 THE SEMIGROUPS $\Gamma_{\lambda+\mu} = \Gamma_\lambda \Gamma_\mu$

Amongst the classes of dilations depending on a positive parameter $\lambda \in \mathbb{R}^+$, those used most often are semigroups of the type

(1.12) $\Gamma_{\lambda+\mu} = \Gamma_\lambda \Gamma_\mu \quad (0 < \langle \lambda, \mu \rangle < \infty) \quad$ and $\quad \Gamma_0 = I$

They have the following properties.

(a) If $\lambda > \mu$ then Γ_λ is invariant under γ_μ. In this case the relation (1.12) is equivalent to $\Gamma_\lambda = \Gamma^\lambda_{\lambda-\mu}\Gamma_\mu$ and, following from Theorem (1.3), $\Gamma_\lambda(X)$ is invariant under γ_μ for all $X \in \mathcal{P}$.

(b) Saying that Γ_λ is extensive is equivalent to saying that the mapping $\lambda \to \Gamma_\lambda$ from $\mathbb{R}^+$ into $\mathcal{P}$ is increasing. $X < \Gamma_\lambda(X)$ implies $\Gamma_\mu(X) < \Gamma_\lambda \cdot \Gamma_\mu(X)$ $= \Gamma_{\lambda+\mu}(X)$; conversely, the implication $\lambda \geq \mu \Rightarrow \Gamma_\lambda > \Gamma_\mu$ has a special consequence for $\mu = 0$, the inclusion $\Gamma_\lambda > I$. Note that property (b) is satisfied when the definition (1.12) is replaced by the following weaker relation:

(1.13) $\Gamma_\lambda \Gamma_\mu < \Gamma_{\lambda+\mu} \quad (\lambda, \mu > 0), \qquad \Gamma_0 = I.$

(c) If the dilation $\Gamma_\lambda : \mathcal{P} \to \mathcal{P}$ is continuous for all $\lambda > 0$ and if $\lambda \downarrow 0$ implies $\Gamma_\lambda \downarrow \Gamma_0 = I$ then the mapping $\lambda \to \Gamma_\lambda : \mathbb{R}^+ \to \mathcal{P}$ is right-continuous. In fact, we find for all λ_0

$$\Gamma_{\lambda_0} = \bigwedge_{\lambda > \lambda_0}\Gamma_\lambda, \qquad \bigwedge_\mu \Gamma_\lambda \Gamma_\mu = \bigwedge_{\lambda > \lambda_0}\Gamma_{\lambda+\mu}.$$

(d) If the family of dilations Γ_λ is increasing w.r.t. λ and if there exists a value $\lambda_0 > 0$ such that Γ_{λ_0} is left-continuous w.r.t. λ_0 then Γ_λ is left-continuous for all $\lambda > \lambda_0$.

(e) If a family of dilations Γ_λ is defined for the domain $[0, 2\epsilon_0]$ and satisfies the semigroup relation for all pairs $(\epsilon, \epsilon') \in [0, \epsilon_0]$ then we can construct an extension of $\mathbb{R}^+$ that retains the semigroup structure (1.12). For each real $\lambda \in \mathbb{R}^+$, let n be its integer quotient by ϵ_0. Set

(1.14) $\Gamma^*_\epsilon = (\Gamma_{\lambda/(n+1)})^{n+1}.$

The dilation Γ^*_ϵ is the iteration of order $n+1$ of a dilation of the family $\{\Gamma_\lambda\}$, since, by construction, $\lambda/(n+1) \in [0, \epsilon_0]$. We then use the commutativity of semigroups to see that $\Gamma^*_\lambda \Gamma^*_{\lambda'} = \Gamma^*_{\lambda+\lambda'}.$

(f) If the dilation Γ_λ, defined in (e) is right-continuous w.r.t. ϵ for $\epsilon \in [0, \epsilon_0]$ then Γ^*_λ is right-continuous for all $\lambda \in \mathbb{R}^+$. Also, if Γ_λ is left-continuous for $\epsilon \in [0, \epsilon_0]$ then Γ^*_ϵ is left-continuous immediately from (1.14).

(g) Let λ be a parameter taking values in the closed half-line $[0, \infty]$, and let Γ_λ be a family of extensive dilations of $\mathcal{P} \to \mathcal{P}$ such that

$$\Gamma_\lambda \Gamma_{\lambda'} < \Gamma_{\lambda+\lambda'} \qquad \text{with } \Gamma_0 = I.$$

The quantities $d(X, Y) = \inf\{\lambda; X < \Gamma_\lambda(Y), Y < \Gamma_\lambda(X)\}$ (if we can find a λ that permits these inclusions) and $d(X, Y) = \infty$ (if we cannot) define a pseudometric on $\mathscr{P}$. The first two axioms for the pseudometric,

$$d(X, Y) = d(Y, X) \quad \text{and} \quad d(X, X) = 0$$

are obvious. For the third, the triangle inequality, let

$$\lambda_1 = d(Y, Z), \qquad \lambda_2 = d(X, Z), \qquad \lambda_3 = d(X, Y).$$

If λ_1 or $\lambda_3 = \infty$ then $\lambda_2 \leq \lambda_1 + \lambda_3$. If not, then from the inclusions

$$X < \{\Gamma_{\lambda_3}(Y) : Y > \lambda_3\} = \Gamma_{\lambda_1+\lambda_3}(X), \qquad Y < \Gamma_{\lambda_1,\lambda_3}(X)$$

and

$$Y < \Gamma_{\lambda_1}(Z), \qquad Z < \Gamma_{\lambda_1}(Y)$$

we find

$$X < \Gamma_{\lambda_3}\Gamma_{\lambda_1}(Z) < \Gamma_{\lambda_1+\lambda_3+2\varepsilon}(X),$$

and also

$$Z < \Gamma_{\lambda_2}\Gamma_{\lambda_1}(X) < \Gamma_{\lambda_1+\lambda_3+2\varepsilon}(X),$$

from which we get $\lambda_2 \leq \lambda_1 + \lambda_3 + 2\varepsilon$ for all $\varepsilon > 0$; therefore

$$\lambda_2 \leq \lambda_1 + \lambda_3.$$

Generally speaking, when the product $\Gamma_\lambda \Gamma_\mu$ is commutative, the first pseudometric leads to an infinity of others, which are ordered amongst themselves. It suffices to take

$$\Gamma_\lambda^{(0)} = \Gamma_\lambda$$

We have $\Gamma_0^{(n)} = I$ and

$$\Gamma_\lambda^{(n+1)} = \Gamma_{\lambda/2}\Gamma_{\lambda/2}\Gamma_{\lambda/2}\Gamma_{\lambda/2} < \Gamma_{\lambda/2}\Gamma_{\lambda/2}\Gamma_{\lambda/2} = \Gamma_\lambda^{(n)}$$

Each iteration of this process gives us a pseudometric that is weaker than the preceding one.

(h) If, beyond the hypothesis of (g), we allow

$$d(X, Y) < \infty \qquad \forall X, Y \in \mathscr{P}$$

and

$$\Gamma_\lambda \downarrow \Gamma_0 = I \quad \text{when} \quad \lambda \downarrow 0$$

then the quantity $d(X, Y) = 0$ It is clear that $d(X, Y) = 0$ is equivalent to

$$X < \wedge\{\Gamma_\lambda(Y); \lambda > 0\} \quad \text{and} \quad Y < \wedge\{\Gamma_\lambda(X); \lambda > 0\};$$

---

that is, because of the hypothesis of right-continuity for Γ_λ, at $\lambda = 0$,

$$X < Y \quad \text{and} \quad Y > X.$$

1.7 EXAMPLES

To illustrate the major theorems of this chapter, we shall again take the three examples of lattices that were introduced in Section 1.1: namely the u.s.c. functions, the partitions and the open (or closed) Euclidean sets.

(a) Lattice $\mathscr{U}$

Amongst the various erosions that we can construct on $\mathscr{U}$, the most frequently used are those called Euclidean (cf. Chapter 5) and defined as follows. Let $\breve{K}$ be a compact set of $\mathbb{R}^n \times \bar{\mathbb{R}}$, let K_x be its translation by the point (x, t), and let $\breve{K}$ be the symmetric set of K with respect to the origin, i.e. $\breve{K} = \{(-x, -t) : (x, t) \in K\}$. Given

$$U(f \ominus \breve{K}) = \{(x, t) : \breve{K}_{x,t} \subset U(f)\}$$

we can easily see that

$$U(f \ominus \breve{K}) = \{(x, t) : \breve{K}_{x,t} \subset U(f)\}$$

Consequently, $U(f \ominus \breve{K})$ is an umbra (and we denote the associated u.s.c. function $f \ominus \breve{K}$), and furthermore the mapping $f \to f \ominus K$ commutes with inf. We therefore have an *erosion* (cf. Fig. 1.3). By applying Theorem 1.6, we see that its dual dilation is given by

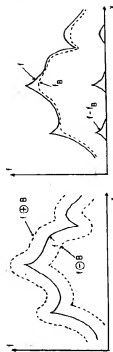$$U(f \oplus K) = \{(x, t) : \breve{K}_{x,t} \cap U(f) \neq \emptyset\} = \bigcup_{(x,t)\in K} [U(f)]_{x,t}$$

Fig. 1.3 (a) Erosion and dilation of function f by disc B. (b) Morphological opening f_B and the difference $f - f_B$ (the latter operation, called the "top-hat transform" (Meyer, 1978) or the "Rolling ball" (Sternberg, 1979), is a contrast algorithm frequently employed in morphology).

(the right-hand side of this equation is a union of closed sets—it is also topologically closed, Matheron, 1975, p. 19). From these expressions, we can deduce the morphological opening and closing. They are traditionally denoted by

$$f_K = (f \ominus \check{B}) \oplus K \quad \text{and} \quad f^K = (f \oplus \check{B}) \ominus K.$$

The set $U(f_K)$ is the zone of space in $\mathbb{R}^n \times \mathbb{R}$ that is swept by the translation of K when they are included in $U(f)$. Here we have a geometrical interpretation of the first relation in (1.5) (there is a dual interpretation for the closing f^K).

The most useful families of operators are those obtained by starting with homothetic sets λK of a compact convex set K. In fact, the dilations Γ_λ defined by

$$\Gamma_\lambda(f) = f \oplus \lambda K \qquad (K \text{ a compact and convex set})$$

satisfy (1.8), thus introducing size distributions, and also satisfy the semigroup relation (1.12). Amongst many relations concerning continuity, we have, for K a compact set,

$$\lambda \uparrow \mu = \begin{cases} f \oplus \lambda K \downarrow f \oplus \mu K, & f^{\lambda K} \downarrow f^{\mu K}, \\ f \ominus \lambda K \uparrow f \ominus \mu K, & f_{\lambda K} \uparrow f_{\mu K}, \end{cases}$$

and

$$f_\lambda \uparrow f = \begin{cases} f_\lambda \oplus K \uparrow f \oplus K, & \bigcup f^K \downarrow f^K, \\ f_\lambda \ominus K \downarrow f \ominus K, & \bigcup f_K \uparrow f_K, \end{cases}$$

These relations can be interpreted as particular cases of another topology, which is more general than that of the present case ($\mathbb{R}^n$ being a topological space) (see Matheron, 1975, p. 16).

(b) Lattice $\mathcal{F}$

These rather formal examples will be complemented with more realistic ones on segmentation in Section 4.8.

Example 1 We define in the lattice $\mathcal{F}$ of partitions of an arbitrary space E the mapping $\dot{\Gamma} : \mathcal{F} \to \mathcal{F}$ as follows:

$$\dot{\Gamma}(T) = \begin{cases} T \wedge T_0 & \text{for all } T \in \mathcal{F}, T \leq T_0, \\ E & \text{if } T = E, \end{cases}$$

where T_0 is a given partition (Fig. 1.4). The operation $\dot{\Gamma}$, which obviously commutes with inf, is an erosion. If $T \leq T_0$, then $\dot{\Gamma}(T) = T$; conversely, if $\dot{\Gamma}(T) = T$ then $T = E$ or $T = T \wedge T_0$ and therefore $T < T_0$. The invariant sets under $\dot{\Gamma}$ form the family $\{T : T < T_0\} \cup E$.

According to Theorem 1.6, the dual dilation Γ is expressed by

$$\Gamma(T) = \wedge \{B : B \in \mathcal{F}, \dot{\Gamma}(B) \geq T\}.$$

which shows that if $T \nleq T_0$ then $\Gamma(T) = T = \dot{\Gamma}(T)$. For $T \nleq T_0$, there exists no set $B \neq E$ such that $\dot{\Gamma}(B) > T$, because we also have $\dot{\Gamma}(B) < T_0$. We then have $\Gamma(T) = E$. In summary,

$$\Gamma(T) = \begin{cases} T & \text{if } T < T_0, \\ E & \text{if } T \nleq T_0. \end{cases}$$

Finally, the morphological closing $\dot{\Gamma}\Gamma$ and opening $\Gamma\dot{\Gamma}$ coincide respectively with dilation and erosion.

Although the opening is too trivial to produce instructive size distributions, the erosion-closing $\dot{\Gamma}$, on the other hand, brings us to antisize distributions. It suffices to parametrize T_0 by $\lambda > 0$ so as to have

$$\lambda > \mu < T_0(\lambda) < T_0(\mu) \quad \text{and} \quad \lambda = 0 \Rightarrow T_0 = E,$$

and then

$$\dot{\Gamma}(T) = T \wedge T_0(\lambda)$$

is an antisize distribution.

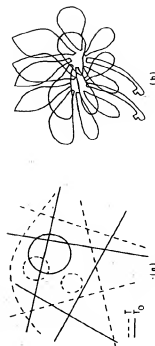$$\Gamma(T) = T \wedge T_h$$

Fig. 1.4 Erosions of partitions: (a) example 1; (b) example 2

Example 2 Now let us consider the lattice $\mathcal{F}$ of partitions of a vector space E. Given a partition $T \in \mathcal{F}$, let T_h denote the partition obtained by translating each class of T by a vector h. Now set

$$\dot{\Gamma}(T) = T \wedge T_h,$$

where h is a given vector. $\dot{\Gamma}$ is clearly an erosion. The corresponding dilations is obtained by noting that $B \wedge B_h > T$ implies $B > T$ and $B_h > T$, i.e. $B > T_{-h}$. The invariant sets under the family $\{T : T < T_{-h}\} \cup E$.

and

$$B > T, \quad B > T_{-x} = B > T \vee T_{-x}$$

but

$$\dot{T}(T \vee T_{-x}) = (T \vee T_{-x}) \wedge (T \vee T_h) > T;$$

therefore

$$\dot{T}(T) = T \vee T_{-x}$$

(From this we can deduce openings and closings, but they present no particular interest.)

Example 3: Hierarchies The concept of indexed hierarchy in a partition was introduced by Simon (1985, p. 125). It is a complete chain in the set of the lattice of partitions, i.e. it begins at ∅ and terminates at E. In terms of mappings, a hierarchy can be interpreted as a succession of transforms of ∅ according to a family $\{\psi_\lambda\}$ of increasing mappings such that

(a) for a $\lambda_0 \geq 0$ we have $\psi_{\lambda_0}(\emptyset) = \emptyset$;
(b) for a $\lambda_1 > \lambda_0$ we have $\psi_{\lambda_1}(\emptyset) = E$;
(c) $\lambda_1 \geq \lambda \geq \mu \geq \lambda_0 = \psi_\lambda(\emptyset) \geq \psi_\mu(\emptyset)$.

It is wise to suppose that the level at which we begin has no affect on the law that controls the hierarchy's progression, i.e. $\lambda > \mu = \psi_\lambda \psi_\mu(\emptyset) = \psi_\lambda(\emptyset)$. Axiom (c) is then replaced by the condition that $\{\psi\}$ be an antizip distribution. We shall see an example of hierarchy in Section 4.8.

(c) Lattice of the open (resp. closed) sets in $\mathbb{R}^n$

Take Euclidean space $\mathbb{R}^n$ of dimension n, and for all $X \subset \mathbb{R}^n$ and all vectors

$h \in \mathbb{R}^n$, let X_h denote the set translation of X by h. We saw in Section 1.1 that the two sets $\mathscr{G}(\mathbb{R}^n)$, which are the topologically open sets, and $\mathscr{F}(\mathbb{R}^n)$, which are the topologically closed sets of $\mathbb{R}^n$, formed two complete lattices.

Consider the Minkowski addition $X \oplus K$ of set X by a structuring element K (see Section 3.2):

$$X \oplus K = \bigcup_{y \in K} X_y, \quad X, Y \subset \mathbb{R}^n.$$

For a given K, $X \oplus K$ is clearly a dilation from $\mathscr{P}(\mathbb{R}^n)$ into itself. The corresponding erosion is written as

$$X \ominus K = \bigcap_{-y \in K} X_{-y}.$$

It has been shown in Matheron (1975, Chapter 1) that if K is compact and X is open then the dilation $X \oplus K$, as well as the corresponding erosion, are open, and the morphological opening and closing are all open. The same is true if K is compact and X is closed, for the four operations that map $\mathscr{F}(\mathbb{R}^n)$ into itself. The reader will find in Section 6.6 an example of the use of lattices $\mathscr{F}$ and $\mathscr{G}$.
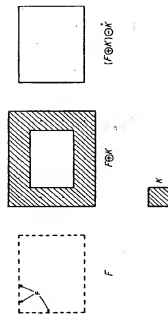

Fig. 1.5 Euclidean dilation and closing of a topologically closed set. (a) Series of segments that outline in dotted line the contour of a rectangle. (b) Dilation by the compact square K. (c) Morphological closing of F.